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POST-CRITICAL BEHAVIOUR OF A LONGITUDINALLY COMPRESSED ROD FOR RIGID LIMITATIONS ON THE DEFLECTION*

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An approach based on the application of optimization methods is developed for determining the state of stress and strain of bodies and structures with given limitation on the displacement. A model problem of plane longitudinal bending of a hinge-supported rod is considered with rigid limitations on the deflection. An analytic solution is obtained for this problem that extends a well-known solution to the non-linear case /1/. Then, by applyiny the Ritz method to a variational problem and replacing the continuous by discrete limitations, the variational problem is transformed into a non-linear programming problem. The results of numerical computations are in good agreement with the analytical solution. A simple proof is given for the complete adjacency hypothesis used to obtain the latter. The mechanism of the formation of the multiwave bending mode as the axial compressive force increases, described in /1/, is confirmed by a numerical experiment.

The problem under consideration is interesting in connection with the need to reveal the stable dynamic bending modes of drilling tube columns in a borehole. One of the methods of solving this problem is based on assumptions about the nature of adjacency of the column to the borehole wall or about the column bending mode. An investigation of the shape of a cambered axis using assumptions of complete adjacency is made in /2/.

We consider the plane bending of a longitudinally compressed rod located initially along the axis of a cylindrical cavity (the radius is $\Delta = \text{const}$) with absolutely rigid walls. Let the hinge-clamped ends of the rod remain on the cavity axis during deformation while the longitudinal compressive force P retains its magnitude and direction. Under such assumptions, the determination of the plane bending mode of the rod reduces to solving the following variational problem

$$\Pi[w] = \int_0^l L(w', w'')\, ds \to \min_{\|w\| \leqslant \Delta} \tag{1}$$

$$w(0) = w(l) = 0, \quad w''(0) = w''(l) = 0$$

$$\left(L(w', w'') = \frac{EI}{2}\frac{w''^2}{1 - w'^2} - P\left(1 - \sqrt{1 - w'^2}\right)\right)$$

where w, w' and w'' are the deflection function, and its first and second derivatives with respect to s, EI is the rod bending stiffness, and l is the rod length.

Furthermore, we assume the force P to be greater than the first critical force $(P > P_*^{(1)} \equiv \pi^2 EI/l^2)$ and greater than the force for which the rod would touch the wall. We assume here that the rod abuts completely on a cavity wall at a certain middle part of length $l_2 = l - 2l_1$ (Fig.1). We call this assumption the hypothesis of total adjacency. When there is a section of total rectification, the determination of the deflection at each of the curvilinear sections (from the hinged end to the first point of tangency) reduces to solving the variational problem

$$\int_0^{l_1} L(w', w'')\, ds \to \min_{w,\ l_1} \tag{2}$$

under the boundary conditions

$$w(0) = w''(0) = 0, \quad w(l_1) = \Delta, \quad w'(l_1) = 0 \tag{3}$$

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The necessary condition for a minimum of (2) in l_1 under the boundary conditions (3) has the form

$$w''(l_1) = 0 \tag{4}$$

If $w''^2 \ll 1$, then the variational problem corresponding to (2)–(4) has the following solution

$$w(s) = \frac{\Delta}{\pi}\sin ks + \frac{k\Delta}{\pi}, \quad k^2 = \frac{P}{EI} \tag{5}$$

Substituting solution (5) into relationship (4) and setting $l_1 = l/2$ we obtain $P = 4\pi^2 EI/l^2$. This means that (by linear theory) when

$$P_*^{(1)} < P \leqslant P_*^{(2)}, \; P_*^{(2)} = 4\pi^2 EI/l^2 \tag{6}$$

the rod can touch the wall at not more than one point and adjacency of the rod to the cavity wall will occur only when $P > P_*^{(2)}$.

Let us investigate the solution of problem (2), (3). The first integral of the Euler equation for functional (2) has the form

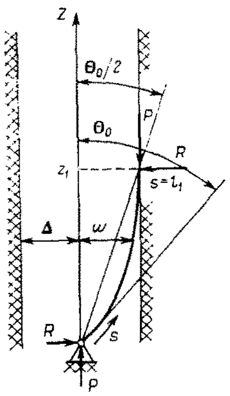

Fig.1

$$\frac{d}{ds}\frac{w''}{1-w'^2} - \frac{w'w''^2}{(1-w'^2)^2} + k^2\frac{w'}{\sqrt{1-w'^2}} =: \tag{7}$$
$$r = \text{const}$$

Taking into account that $w' = \sin\theta$ (θ is the slope of the tangent to the rod axis), the first integral of (7) can be written thus:

$$\begin{gathered} {}^1/_2\theta'^2 = A\cos(\theta+\beta) + C \\ A = \sqrt{k^4 + r^2}, \; \sin\beta = -r/A, \\ \cos\beta = k^2/A \end{gathered} \tag{8}$$

Because of conditions (3), $\theta'(0) = 0$, consequently, $C = A\cos(\theta_0+\beta)$ and (8) reduces to the form

$$\frac{1}{2}\theta'^2 = 2A\left(\sin^2\frac{\theta_0+\beta}{2} - \sin^2\frac{\theta+\beta}{2}\right), \quad \beta = -\frac{\theta_0}{2} \tag{9}$$

where it is taken into account that $\theta(l_1) = 0, \theta'(l_1) = 0$ (see (3) and (4)). Furthermore, making the Euler substitution

$$\sin\frac{\theta+\beta}{2} = -\sin\frac{\theta_0+\beta}{2}\sin\psi \tag{10}$$

and taking account of the condition $\psi_0 = \psi(0) = -\pi/2, \psi_1 = \psi(l_1) = \pi/2$, resulting from (10), we obtain

$$ks = \cos^{1/2}\frac{\theta_0}{2}[F(m,\psi) + \mathbf{K}(m)]; \quad m^2 = \sin^2\frac{\theta_0}{4} \tag{11}$$

($F(m,\psi)$ is the elliptic integral of the first kind, $\mathbf{K}(m) = F(m, \pi/2)$).

In particular, we obtain from (11) a formula for the length of the curvilinear section

$$kl_1 = 2\cos^{1/2}\frac{\theta_0}{2}\mathbf{K}(m) \tag{12}$$

Taking the notation of (10) into account, we find

$$\begin{gathered} kw(\psi) = k\int_0^s \sin\theta\, ds = \cos^{1/2}\frac{\theta_0}{2}\left\{\sin\frac{\theta_0}{2}[2\mathbf{E}(m) - \mathbf{K}(m)] + \right. \\ \left. \sin\frac{\theta_0}{2}[2E(m,\psi) - F(m,\psi)] + 2\cos\frac{\theta_0}{2}\sin\frac{\theta_0}{4}\cos\psi\right\} \end{gathered} \tag{13}$$

($E(m,\psi)$ is the elliptic integral of the second kind, $E(m) = E(m, \pi/2)$).

The following fundamental equation of the problem results from (13)

$$k\Delta(\theta_0) = 2\cos^{1/2}\frac{\theta_0}{2}\sin\frac{\theta_0}{2}[2\mathbf{E}(m) - \mathbf{K}(m)] \tag{14}$$

We show in Fig.2 a graph of the function $k\Delta(\theta_0)$ from which it is seen that two assumed equilibrium modes correspond to each value of the criterion parameter $k\Delta = \Delta\sqrt{P/(EI)}$ while $k\Delta < \max k\Delta(\theta_0) \approx 1.66$, $\theta_0 \in [0, \pi]$. A graph of the function $kl_1(\theta_0)$ is also represented there.

Using $z' = \cos\theta$ as the initial relationship and carrying out calculations analogous to those made in deriving (13), we obtain

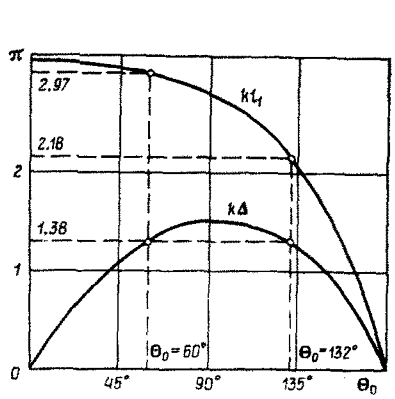

Fig.2

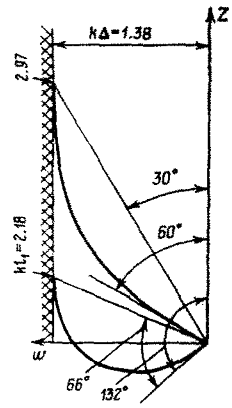

Fig.3

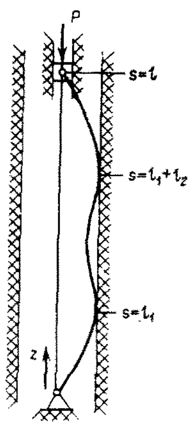

Fig.4

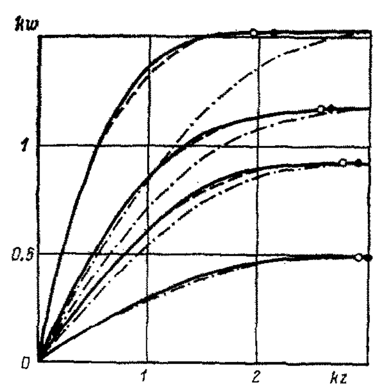

Fig.5

$$kz(\psi) = \cos^{1/2}\frac{\theta_0}{2}\left\{\cos\frac{\theta_0}{2}\left[2\mathbf{E}(m) - \mathbf{K}(m)\right] + \cos\frac{\theta_0}{2}\left[2E(m,\ \psi) - F(m,\ \psi)\right] + 2\sin\frac{\theta_0}{2}\sin\frac{\theta_0}{4}\cos\psi\right\} \tag{15}$$

Setting $\psi = \pi/2$ in (15) and taking into account (14), we arrive at the relationship $(z_1 = z(\pi/2))$

$$\Delta = z_1 \operatorname{tg}(\theta_0/2) \tag{16}$$

which means that the beginning of the lining section is at the intersection of the bisector of the angle θ_0 with the boundary wall (see Figs.1 and 3).

The necessary conditions for the deflection mode with a lining to exist are written as follows:

$kl > \pi$ or $P > P_*^{(1)}$ is the condition for the possibility of a non-rectilinear equilibrium mode to exist for a rod of length l;

$kl_2 > 0$ is the condition for the lining section to exist;

$kl_2 < 2\pi$ is the stability condition for the lining section

$$(kl_2 \equiv kl - 4\mathbf{K}\cos^{1/2}\theta_0/2)$$

It hence follows that tangency is possible for

$$P = \nu^2\pi^2\frac{EI}{l^2}\left(\nu = \frac{4\mathbf{K}}{\pi}\cos^{1/2}\frac{\theta_0}{2}\right) \tag{17}$$

It is seen that as θ_0 increases the value of P corresponding to the beginning of the lining is reduced from $P = P_*^{(2)}$ (for $\theta_0 \ll 1$) to $P = P_*^{(1)}$ (for $\theta_0 \approx 156°$).

Remark. As in /1/, the assumption was made above about the presence of a lining section (the hypothesis of total adjacency). We will give a proof of the existence of such a section. Let the rod have the equilibrium mode shown in Fig.4. At the sites of contact with the wall the rod cannot have lining sections. Otherwise, the problem for the middle section would turn out to be overdefined: at the end of the rod it would be necessary to satisfy three conditions $w = w' = w'' = 0$. We select an imaginary middle section of the rod and we connect the outer sections into one rod. We then obtain a cambered equilibrium mode for a hinge-supported rod of length $2l_1$. This equilibrium mode can hold only for $2kl_1 > \pi$. The middle section is the cambered equilibrium mode of a rod of length l_2 clamped rigidly at the ends. Such an equilibrium mode is possible for $kl_2 > 2\pi$. We hence obtain

$$k(2l_1 + l_2) \equiv kl > 3\pi$$

Therefore, the equilibrium mode under consideration is possible only for $P > P_*^{(3)}$. This means that for any θ_0 between the second and third critical forces of a rod of length l $(P_*^{(2)} < P < P_*^{(3)})$ only one equilibrium mode is possible for the middle section, namely lining.

Example. Let $l = 1$ m, $EI = 10^{-3}$ kN.m^2, $\Delta = 0{,}11$ m. Then we have $k\Delta = \Delta\sqrt{P/(EI)} = 1.38$ for $P = P_*^{(4)} = 0.158$ kN. From the graph in Fig.2 we find $(\theta_0)_1 = 60°$, $(\theta_0)_2 = 132°$. Corresponding to these values of θ_0 are $(kl_1)_1 = 2.97$ and $(kl_1)_2 = 2.18$ or $(l_1)_1 = 0.174$ m and $(l_1)_2 = 0.236$ m. The lengths of the corresponding lining sections are $(l_2)_1 = 0.652$ m and $(l_2)_2 = 0.528$ m (Fig.3).

Numerical solution of problem (1). The hinge-support boundary conditions will be satisfied if the deflection function is sought in the form

$$w(s) \approx w(x;s) = x_n \sin\frac{\pi s}{l} + \left(\frac{2}{l}\right)^6 (s^2 - ls)^3 \sum_{i=0}^{r-2} x_{i+1} \left(\frac{2}{l}\right)^{2i} \left(s - \frac{l}{2}\right)^{2i} \tag{18}$$

(Note that in the numerical solution of problem (1) the deflection function was initially sought in the form of a partial sum of a trigonometric series in the eigenfunctions of the linear problem of longitudinal bending $w(s) = \sum_{i=1}^{r} x_i \sin i\pi s/l$. In such an approximation of the deflection function the bending mode with a lining section turned out to be unstable in calculational respects, which indeed necessitated consideration of a polynomial representation of the desired function.)

Substituting $w(s)$ defined by (18) into the functional (1), we obtain

$$f(x) = \int_0^l L[w_s'(x;s), w_{ss}''(x;s)]\,ds \to \min_x \tag{19}$$
$$h(x;s) \equiv |w(x;s)| - \Delta \leqslant 0$$

Replacing the continuous constraint in (19) by a set of discrete constraints, we arrive at the following non-linear programming problem

$$f(x) \to \min_{x \in \Omega} \tag{20}$$
$$\Omega = \{x \in E_n \mid h_j(x) \leqslant 0,\ j \in 0:N_1\}$$
$$h_j(x) = h(x; s_j),\ s_j = jl/N_1$$

The function $f(x)$ and its gradient were evaluated by a Gauss quadrature formula with 32 nodes. Problem (20) was solved by the method of ε-steepest descent /3/ for $n = 15$, $N_1 = 50$. A feature of the bending mode found by using non-linear programming is the stand-off of the rod from the wall in that part where the lining section should theoretically be. However, as the accuracy of the solution of the non-linear programming problem increases (which is equivalent to an increase in computing time) a tendency to rectification of the middle section is observed. Therefore, the numerical solution is also in agreement with the total adjacency hypothesis.

Graphs of the deflection function are presented in Fig.5 for $P = 0.97\,P_*^{(3)}$ and $k = 9.29$ m^{-1} for different values of Δ. The solid line with the dark point is the analytical solution, the dash-dot curve is the solution in /1/. Comparing these solutions we arrive at the deduction that on being vien a 5% error, linear theory can be used in calculations for $\Delta/l \leqslant 0.06$.

The evolution of deflection was also investigated as the load increased. It is disclosed that a change in the equilibrium mode occurs for a force greater than $P_*^{(4)}$ in the form of a half-wave in a mode with three half-waves. Therefore, a numerical experiment has confirmed the mechanism of multiwave bending mode formation as the axial compressive force increases, as described in /1/.

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EVOLUTION OF A WEAK SIGNAL IN A MAGNETIC MATERIAL*

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The variation of a weak arbitrary perturbation of the magnetic field with time in a magnetic material is investigated. It is assumed that the cubic relation connecting the strength H of the magnetic field with its induction B holds for the material in question. Such a relationship represents a special case of the relation between these quantities for the initial magnetization of the majority of magnetic materials /1/. It is shown that after a certain finite time the signal degenerates into a simple wave and a closing trailing shock. The profile area remains constant and equal to the area of the initial signal.

The formation of shock waves at the front and the slope of the electromagnetic waves was studied earlier in /2, 3/ for a magnet magnetized to saturation, and for the case of precession of the constant magnetization vector.

Let a weak signal $B(x, t_0)$, be generated in the magnetic material at the instant t_0, moving over zero background in the positive direction of the x axis (Fig.1). We shall consider the variation in the given profile with time for the case when the relation connecting H and B has the form

$$H = a_0(B + B^*) + \gamma(B - B_0)^3 \tag{1}$$

Here a_0, γ are certain constants and $a_0 \gg \gamma B_0^2$; B_0 is the value of the magnetic induction for which $d^2H/dB^2 = 0$; B^* is chosen so that when $B = 0$, H is also zero.

The evolution of a weak signal in a magnetic material was investigated in /4/ for the case of a quadratic dependence of H on B.

When the relation $H(B)$ is given, the velocity of small perturbations $u = c\sqrt{a_0}$ is nearly equal to the velocity of the resulting discontinuities /5/

$$v_p = c\{a_0 + \gamma[(B - B_0)^2 + (B - B_0)(B_1 - B_0) + (B_1 - B_0)^2]\}^{1/2}$$

Therefore the perturbations reflected from the discontinuity can be neglected and the initially specified signal will propagate in one direction (e.g. to the right) in the form of a simple wave, with the signal area preserved.

$$s = \int_{x_1}^{x_2} B\,dx$$

Since the rate of propagation of simple waves $a = c\sqrt{dH/dB}$ depends on B, different points of the profile will move through the magnetic material with different velocities, and this will lead to signal distortion, non-uniqueness of the solution, and the formation of discontinuities.

When the signal profile is deformed, the area s_1 corresponding to the extension of the shock zone will be equal to the area s_2 which would be traversed over the same time by a simple wave in the case of non-uniqueness. Indeed,

$$s_1 = v(B - B_1) = c\left(\frac{H - H_1}{B - B_1}\right)^{1/2}(B - B_1) =$$
$$c\left[\frac{a_0(B - B_1) + \gamma(B - B_0)^3 - \gamma(B - B_1)^3}{B - B_1}\right]^{1/2}(B - B_1) \approx$$
$$ca_0^{1/2} + \frac{\gamma c}{2a_0^{1/2}}\left[(B - B_0)^3 - (B_1 - B_0)^3 \approx\right.$$
$$s_2 = \int_{B_1}^{B_2} a\,dB = \int_{B_1}^{B_2} c\left(\frac{dH}{dB}\right)^{1/2} = c\int_{B_1}^{B_2}[a_0 + 3\gamma(B - B_0)^2]^{1/2}\,dB$$

*Prikl.Matem.Mekhan.,49,1,161-163,1985